“If you want to know the true character of a person, divide an inheritance with him.” Ben Franklin

Example:
Divide 20 pieces of candy among 4 equally deserving children.

\[ \frac{20}{4} = 5 \text{ pieces to each} \]

What if the candy is not all the same?

EQUAL # of pieces may not be a fair share.

The big question in this chapter: How can we divide up a set of diverse goods so that everyone walks away feeling as though they got a fair share?

Why this is important: Land rights, Countries (like the former Yugoslavia), dividing the rights for oceanic mining, dividing global responsibility for the environment, wills & estates, divorce proceedings... etc.

3.1 Fair Division Games

Elements:

- The **GOODS**: typically tangible, physical objects that will be divided. Symbol used: \( S \) for the total; \( S_1, S_2, \) etc for the pieces

- The **Players**: the parties involved who each have a right to a share of the goods. Symbol used: \( P_1, P_2, P_3, \) etc.

- The **Value** systems: each player has the ability to look at the set of goods and assign a value such as “to me this is worth \( \$4.00 \)” or “to me this is worth \( 30\% \) of the total.”

Assumptions:

- **Cooperation** – everyone is willing to participate and accept the rules.

- **Rationality** – everyone acts on reason alone, seeking to maximize their share of the goods.

- **Privacy** – no player has any useful info on any other player’s values

- **Symmetry** – players have **EQUAL RIGHTS** in sharing the goods.

No player more important than any other
Fair Division Scheme or Method: A set of **RULES** defining how our game is to be played

Type 1: **Continuous** fair division game = goods can be divided repeatedly

Think: **Money, Land, Cake**.

Type 2: **Discrete** fair division game = goods cannot be divided

Think: **Cars, Houses, People**.

Outcome/GOAL: Everyone feels like they got a **Fair Share** of the goods.

So what does it mean to get a **fair share**?

\[ \begin{align*}
N &= \# \text{ players} \\
S &= \text{set of goods} \\
P &= \text{one of the players}
\end{align*} \]

A fair share to player \( P \) =

a portion of \( S \) worth at least \( \frac{1}{N} \)th of the total value of \( S \) in the **opinion** of player \( P \).

For example: If there are 5 players, a fair share would be worth at least \( \frac{\sqrt{5}}{5} \) of the total in the eyes of the player receiving it. (or 20%)

**Finding Fair Shares:**

**Example A.1**

Mrs. Leahy buys a \( \frac{1}{2} \) chocolate, \( \frac{1}{3} \) vanilla cake for $20. Mrs. Leahy values chocolate three times as much as vanilla.

How much is the chocolate half worth to her? How much is the vanilla half worth?

\[ \begin{align*}
x + 3x &= $20 \\
4x &= $20 \\
x &= $5
\end{align*} \]

\[ \begin{align*}
\text{chocolate} &= $15 \\
\text{vanilla} &= $5
\end{align*} \]

Notice there are 180° of vanilla and 180° of chocolate.
(example A1 continued)
If the chocolate half is worth \$15 and the vanilla half is worth \$5 to Mrs. Leahy,

How much is this slice worth to Mrs. Leahy?

\[
\text{value vanilla} \quad \downarrow \quad \frac{60^\circ}{180^\circ} (\$5) = \boxed{\$1.67}
\]

How much is this slice of cake worth to Mrs. Leahy?

\[
\text{value vanilla} \quad \downarrow \quad \frac{30^\circ}{180^\circ} (\$5) + \frac{20^\circ}{180^\circ} (\$15) = 0.83 + 1.67 = \boxed{\$2.50}
\]

If Mrs. Leahy is splitting this cake with 4 friends, how much would a fair share be worth?

5 players \( \frac{1}{5} \text{(total)} = \frac{1}{5} \times 20 = \boxed{\$4.00} \) \leftarrow \text{fair share}

Example A.2.
Mr. Record buys a cake that is 1/3 of each vanilla, chocolate and strawberry. He values vanilla twice as much as chocolate and strawberry twice as much as vanilla. If the whole cake costs \$14.00, how much is each individual flavor worth?

\[
x + 2x + 4x = 14.00
\]

\[
7x = 14
\]

\[
x = 2
\]

Chocolate = \$2.00
Vanilla = \$4.00
Strawberry = \$8.00

Notice: \( 360^\circ \div 3 = 120^\circ \) each flavor

How much is this slice of cake worth to Mr. Record?

\[
\text{value strawb.} \quad \downarrow \quad \frac{30^\circ}{120^\circ} (8.00) + \frac{30^\circ}{120^\circ} (2.00) = 2 + 0.5 = \boxed{\$2.50}
\]
Example A3: Three players (Abby, Betty, and Cindy) are dividing a pizza into three slices: s1, s2, s3. The following table shows the value of each slice in the eyes of the player. Which slices are fair shares to each player?

\[
\frac{1}{3} = 33\frac{1}{3}\% = \text{fair share}
\]

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>32%</td>
<td>33%</td>
<td>35%</td>
</tr>
<tr>
<td>B</td>
<td>34%</td>
<td>42%</td>
<td>24%</td>
</tr>
<tr>
<td>C</td>
<td>35%</td>
<td>34%</td>
<td>31%</td>
</tr>
</tbody>
</table>

Fair share \(\geq 33\frac{1}{3}\%\)

A: s3
B: s1 and s2
C: s1 and s2

\[\begin{array}{ccc|c}
\text{total} & S1 & S2 & S3 \\
A & \$5.00 & \$6.00 & \$4.00 \\
B & \$3.99 & \$4.00 & \$4.01 \\
C & \$7.50 & \$6.50 & \$7.00 \\
\hline
= 15.00 & = 12.00 & = 21.00 \\
\end{array}\]

\[\frac{\text{total}}{3} = \text{Fair Share}\]

\[\frac{15.00}{3} = 5.00 \quad \frac{12.00}{3} = 4.00 \quad \frac{21.00}{3} = 7.00\]

A: any share \(\geq 5.00\)
S1, S2

B: any share \(\geq 4.00\)
S2, S3

C: any share \(\geq 7.00\)
S1, S3

Is there a way to assign pieces to everyone so that each player feels like they got a fair share?

A = s3
B = s1 or s2
C = s2

3.2 Two Players: The Divider – Chooser Method

You cut— I choose.
Cake:
Player 1: likes chocolate and strawberry equally
Player 2: only likes strawberry

Step 1: Divider (P1) cuts the cake into two pieces S1 & S2
Step 2: Chooser (P2) picks the “best” piece
Step 3: Divider gets the remaining piece.

It is always more preferable to be the Chooser!

Using Divider-Chooser, the divider splits the goods into two equal pieces in his opinion. So he will always end up with a portion worth exactly \(50\%\) to him.

The chooser gets to pick the best piece in his opinion and will frequently have the chance to pick a piece worth \(> 50\%\) in his opinion.
Example B.1. Two friends are going to divide a four flavor cake: chocolate, orange, strawberry, and vanilla. Arthur likes chocolate, orange, and strawberry equally well, but hates vanilla. Brian likes chocolate and strawberry equally well, but hates orange and vanilla.

Suppose Arthur makes the first cut. Which of the following cuts are possible 50/50 cuts to Arthur? For each cut that is consistent with Arthur’s value system, which piece would Brian choose?

\[ S_1 = \frac{180^\circ}{360^\circ} = 50\% \]
\[ S_2 = \frac{90^\circ}{360^\circ} = 25\% \]

\[ S_1 = \frac{135^\circ}{270^\circ} = 50\% \]
\[ S_2 = \frac{35^\circ}{270^\circ} = 13\% \]

Brian would choose \( S_1 \)

\[ S_1 = \frac{135^\circ}{180^\circ} = 75\% \]
\[ S_2 = \frac{45^\circ}{180^\circ} = 25\% \]
Example B.2.
Ann values vanilla cake 4 times as much as chocolate.
Bob values chocolate cake three times as much as vanilla cake.

\[\begin{align*}
\text{Ann:} & \quad \frac{4x}{x} = \frac{80}{20} \\
4x + x &= 100 \% \\
5x &= 100 \% \\
x &= 20 \% \\
\text{Bob:} & \quad \frac{3x}{x} = \frac{25}{75} \\
x + 3x &= 100 \% \\
4x &= 100 \% \\
x &= 25 \%
\end{align*}\]

Which of the following cuts are possible 50-50 cuts to Ann?
Which piece would Bob choose and what would it be worth to him?

Bob could choose either
Worth 50\%
**Example B3:** Suppose two friends are going to divide a tub of $12.00$ tub of Neapolitan ice cream (strawberry, vanilla, chocolate using the divider chooser method. Albert likes vanilla and strawberry the same, but likes chocolate 4 times as much as strawberry. Victoria likes vanilla and chocolate the same but hates strawberry and will not eat it.

Assume Victoria and Albert are new friends and do not understand each other’s preferences.

Describe how Albert might cut the ice cream into two 50-50 slices. Which piece would Victoria choose? What would the piece be worth to her?

A: \[
\begin{align*}
\text{Strawberry:} & \quad x \\
\text{Vanilla:} & \quad x \\
\text{Chocolate:} & \quad 4x \\
\end{align*}
\]

\[
\begin{align*}
6x &= 12 \\
x &= 2 \\
S &= \$2 \quad V = \$2 \quad C = \$8
\end{align*}
\]

Fair share = $6

\[
\frac{x}{4} = \frac{6}{8} = \frac{3}{4} \text{ "squares" of chocolate}
\]

\[
\begin{align*}
x &= 3 \\
\text{worth } \$6.00
\end{align*}
\]

Victoria "sees"

\[
\begin{align*}
S_1 &= 5 \text{ squares like } \$2 \text{ worth } \$6.00 \\
S_2 &= 3 \text{ squares like } \$8 \text{ worth } \$12.00
\end{align*}
\]

Victoria choose $S_1$ worth $\frac{5}{8} (62.5\%)$ or $\$7.50$

Describe how Victoria might cut the ice cream into two 50-50 slices. Which piece would Albert choose? What would the piece be worth to him?

\[
\begin{align*}
\text{Strawberry:} & \quad x \\
\text{Vanilla:} & \quad x \\
\text{Chocolate:} & \quad C
\end{align*}
\]

\[
\begin{align*}
2x &= 12.00 \\
X &= \$6.00 \\
S &= \$0 \quad V = \$6 \quad C = \$6
\end{align*}
\]

Albert "sees"

\[
\begin{align*}
S_1 &= \$2 + \$4 = \$6 \\
S_2 &= \$8
\end{align*}
\]

Albert will choose $S_2$ worth $\$8.00
3.3 The Lone Divider Method
(3 or more players)

First, we have to choose a divider. Since it is always more favorable to be a chooser than a divider, we do this by rolling a die, drawing for the high card in a deck of cards, or some other game of chance.

**Step 1: Dividing**
Divider "D" cuts cake into N "equal to him" pieces (each having a value of 1/N to him)

**Step 2: Bidding**
Choosers write down (on a secret ballot) ALL pieces they consider to be a fair share.

**Step 3: Distribution**
Ballots are opened and the pieces are divided so that everyone gets a piece off their list of bids.

**Case 1:** There is a way to assign each player a piece off his or her list. The divider gets the remaining piece.

**Example C1:**

Three friends choose to split a cake using the lone divider method. The players value the pieces as shown in the chart at the right.

Assuming all players play honestly, what would C1 and C2’s bids be? Describe a possible fair division of the cake.

\[
FAIR = \frac{1}{3} = 33\frac{1}{3}\% 
\]

**Case 2:** A Standoff
Two choosers only want the same piece or three choosers want the same two pieces. Assign remaining choosers non “standoff” pieces. Divider chooses one of the unwanted pieces. The remaining pieces are combined into one piece. Standoff choosers use divider-chooser method.

**Example C2:**
Three friends choose to split a cake using the lone divider method. The players value the pieces as shown in the chart at the right.

Assuming all players play honestly, what would C1 and C2’s bids be? Describe a possible fair division of the cake.

\[
Fair = \frac{1}{3} = 33\frac{1}{3}\% 
\]
For Example C3 and C4, Four players decide to split a cake using the lone-divider method. The Divider splits the cake into four shares: S1, S2, S3, and S4. The choosers C1, C2, C3 value the shares as shown in the charts below.

Assuming each player bids honestly, describe each player’s bid. Then describe a possible fair division of the cake.

**Example C.3**

Fair = ¼ = 25%

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>30%</td>
<td>20%</td>
<td>35%</td>
<td>15%</td>
</tr>
<tr>
<td>C2</td>
<td>20%</td>
<td>20%</td>
<td>40%</td>
<td>20%</td>
</tr>
<tr>
<td>C3</td>
<td>25%</td>
<td>20%</td>
<td>20%</td>
<td>35%</td>
</tr>
</tbody>
</table>

C1: S1, S3, S4
C2: S3
C3: S1, S4, S3

**bids**

**fair division**

C1: S1
C2: S3
D: S2

**Example C.4.**

Fair = ¼ = 25%

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>20%</td>
<td>20%</td>
<td>20%</td>
<td>40%</td>
</tr>
<tr>
<td>C2</td>
<td>15%</td>
<td>35%</td>
<td>30%</td>
<td>20%</td>
</tr>
<tr>
<td>C3</td>
<td>22%</td>
<td>23%</td>
<td>20%</td>
<td>35%</td>
</tr>
</tbody>
</table>

C1 = S4
C2 = S2
C3 = S3

**bids**

**fair division**

D = S1
C1 + C3 combine
S3 + S4 then use divider chooser

C2: S2

**Example C.5.** Describe a possible fair division of the cake given the following bids:

A {s1, s2}  
B {s1}  
C {s1, s2, s3, s4}  
D {s2, s3, s4}  

Who was the divider? [C] (feels all 4 pieces are fair shares)

A: S2  
B: S1  
C: S4  
D: S3

or

A = S2  
B = S1  
C = S3  
D = S4
Example C6: Four partners want to divide a parcel of land worth $120 thousand using the lone-divider method. Using a map, the divider cuts the land into four parcels ($s_1, s_2, s_3, s_4$) and the players make their bids.

The value of each parcel (in thousands of dollars) in each chooser’s eyes is given in the following figure. Some of the bids are missing.

What is a fair share worth?

\[ \$120 \div 4 = \$30 \text{ thousand} \]

Who was the divider? Explain. Pete All 4 shares “Fair” = $30 thousand

Determine each chooser’s bid.

A: \( s_3, s_4 \)  
C: \( s_3 \)  
P: divider  
M: \( s_1, s_3, s_4 \)

Find a fair division of the land.

A: \( s_4 \)  
C: \( s_3 \)  
P: \( s_2 \)  
M: \( s_1 \)
3.4 The Lone Chooser Method

Basic Strategy: Divider 1 cuts cake into two “equal pieces” S1 and S2 (50%-50%)
Divider 2 chooses either S1 or S2
Each Divider splits their half into 3 “equal” pieces A, B, and C (33.3%, 33.3%, 33.3%)
Chooser picks one third from Divider 1 and Divider 2.

Step 1: Division
D1 and D2 divide the goods using Divider/Chooser method

Step 2: Sub division
Each divider sub-divides his share into 3 “equal” pieces

Step 3: Distribution
Chooser selects 1 share from each divider

Example
C: takes $1A$ from D1
    takes $2B$ from D2
D1: keeps $1B + 1C$
D2: keeps $2A + 2C$

Example D.1: (#33 pg 109) Angela, Boris, and Carlos decide to divide a vanilla-strawberry cake using the lone choose method. The players value the cake as shown

Suppose Angela and Boris are the dividers and Carlos is the chooser. In the first division, Angela cuts the cake vertically down the center and Boris takes the right half.

A) Describe how Angela would subdivide her share into three pieces. What would each piece be worth to Carlos? Which would he take?

\[
\begin{align*}
\$18 &= \frac{4.50}{13.50} \\
3 \text{ equal } &= \frac{18}{3} = \$6.00 \\
\theta^\circ &= \frac{6.00}{90^\circ} = 13.50 \\
13.5 \times &= 540 \\
x &= 40^\circ
\end{align*}
\]
B) Describe how Boris would subdivide his share into three pieces. What would each be worth to Carlos? Which would he take?

\[
\begin{align*}
3 \text{ equal } & \frac{15}{3} = \$1 \\
\frac{x^\circ}{90^\circ} &= \frac{5}{9.00} \\
x &= \frac{450}{9.00} \approx 50^\circ \text{ vanilla} \\
\frac{x^\circ}{90^\circ} &= \frac{5}{9} \\
x &= \frac{450}{9} \approx 50^\circ \text{ strawberry}
\end{align*}
\]

Carlos
\[
\begin{align*}
2A &= \frac{75^\circ (24)}{180} \\
2B &= \frac{15^\circ (24) + 40^\circ (12)}{180} \\
2C &= \frac{50^\circ (12)}{180} = 3.33
\end{align*}
\]

C) Describe a possible final fair division of the cake.

Carlos: 1A 10^\circ V + 90^\circ S (worth 12.67)
2A 75^\circ S (worth 10.00)

Angela: 1B 40^\circ V (worth 6.00)
1C 40^\circ V (worth 6.00)

Boris: 2B 15^\circ S + 40^\circ V (worth 5.00)
2C 50^\circ V (worth 5.00)

D) What is the value of each piece in the eyes of the player receiving it?

Carlos = $22.67
Angela = $12.00
Boris = $10.00

Note:
Carlos (24+12)/3 = $12 = fair share
Angela (9+27)/3 = $12 = fair share
Boris (12+18)/3 = $10 = fair share
Example D.2: (#37 p110)

Three players: Arthur, Brian, and Carl decide to divide a chocolate-strawberry-vanilla-orange cake using the lone chooser method.

Arthur likes chocolate and orange, but hates strawberry and vanilla.
Brian likes chocolate and strawberry, but hates orange and vanilla.
Carl likes chocolate and vanilla, but hates orange and strawberry.

Suppose Arthur and Brian are the dividers and Carl is the Chooser.

a) Arthur makes the first cut down the center and Brian chooses the piece he likes best.
Which piece would Brian choose? Describe how he might subdivide it into 3 shares.

b) Describe how Arthur might subdivide the other share.

c) Describe a possible fair division of the cake.

d) Find the value of the share (as a percentage of the total cake) in the eyes of the player receiving it.

a)

Brian "sees"
\[ S_1 = 100\% \]
\[ S_2 = 0\% \]
Choose \( S_1 \)

\[ 180^\circ \div 3 = 60^\circ \]
\[ 100\% \div 3 = 33\frac{1}{3}\% \]

b)

\[ 90^\circ \div 3 = 30^\circ \]
\[ 50\% \div 3 = 16\frac{2}{3}\% \]

\[ 1A + 2C \rightarrow \frac{60^\circ + 30^\circ + 90^\circ}{90^\circ} \]
\[ \frac{60^\circ (50\%)}{90^\circ} + \frac{30^\circ (0\%)}{90^\circ} + \frac{90^\circ (50\%)}{90^\circ} \]
\[ 33.3\% + 0\% + 50\% \]
\[ = 83.3\% \]

b)

\[ 30^\circ + 90^\circ \]
\[ \frac{30^\circ (50\%)}{90^\circ} + \frac{90^\circ (50\%)}{90^\circ} \]
\[ 16.6 + 50\% \]
\[ = 66.6\% \]

Arthur: \[ 2A + 2B = 60^\circ \]
\[ \frac{60^\circ (50\%)}{90^\circ} = 33\frac{1}{3}\% \]
3.5 The Last-Diminisher Method:

How this works:

Preliminaries: Players are assigned to play in a fixed order using some game of chance.
P1 goes first, P2 goes second, etc.

The game is played in rounds. During each round “S” (the set of goods) will be in two pieces –

\[ C = \text{the piece that is currently being claimed} \quad \text{or} \quad R = \text{the rest of S} \]

P1: “cuts” a piece off of “S” that he considers to be a fair share

(1/Nth of the total). We call this the C-piece.
P1 is called the “claimant”.

P2: Choice – Pass or Diminish

Pass—If P2 thinks C is less than or equal to 1/N he passes and P1 remains the Claimant of the C-Piece.

Diminish – If P2 think C is more than 1/N, he takes the piece away from P1 and trims it down to a fair share. C is now smaller. The trimmed part goes back to R. P2 is now the claimant of C.

P3: Choice – Pass or Diminish

Each player has a choice to Pass if they think the C-piece is 1/N or less or Diminish if they think the C-piece is worth more than 1/N.

The LAST DIMINISHER (the current claimant) gets to keep C as his share and is out of the game.

The remaining players move on and repeat the process to divide R until everyone has a share.

The last two players use Divider-Chooser.

***** POWERPOINT ***** “The Castaways”
Example E1:

Five players (P1, P2, P3, P4, P5) decide to divide a $40 cake using the last-diminisher method. In Round 1, P1 makes the first cut and makes a claim on a C-piece. The following table gives the value of the current C-piece in the eyes of the other players at the time that it is that player’s turn to play.

<table>
<thead>
<tr>
<th>Value of C Piece</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
<th>P5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6.50</td>
<td>$8.50</td>
<td>$8.20</td>
<td>$7.90</td>
<td></td>
</tr>
</tbody>
</table>

\[
\$40 \div 5 = \$ 8 \text{ fair share}
\]

a) Which players are diminishers in round 1? P3, P4, b/c they see c-piece as > $8.00

b) Who gets a fair share in the end of round 1? What is the value of the share to the player?

P4 (the last diminisher) $8.00

c) In round 2: P1 makes the first cut.

The R piece is worth $32.00 to each of the remaining players.

\[
\$40 - 8 = \$32
\]

Example E2:

Seven players (P1, P2, etc) decide to divide a $210,000 parcel of land using the last-diminisher method. In Round 1, P1 makes the first cut and makes a claim on a C-piece. The following table gives the value of the current C-piece in the eyes of the other players at the time that it is that player’s turn to play.

<table>
<thead>
<tr>
<th>Value of C Piece</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
<th>P5</th>
<th>P6</th>
<th>P7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$31,000</td>
<td>$33,000</td>
<td>$28,000</td>
<td>$29,000</td>
<td>$32,000</td>
<td>$33,000</td>
<td></td>
</tr>
</tbody>
</table>

\[
210,000 \div 7 = 30,000 \text{ fair share}
\]

a) Which players are diminishers in round 1?

P2, P3, P4, P7

b) Who gets a fair share in the end of round 1? What is the value of the share to the player?

P7 (the last diminisher) * Note * below

\[
\$33,000
\]

To any other player, they end up w/ exactly $30,000

P2

c) In round 2: P1 makes the first cut.

The R piece is worth $180,000 to each of the remaining players.

\[
210,000 - 30,000 = 180,000
\]

*** Note: ***

If the LAST DIMINISHER is the LAST PLAYER in a round they are at a special advantage. No player is coming after them, so instead of diminishing to EXACTLY 1/N, they should diminish by as little as possible. Essentially they can choose to diminish by 0%.
Example E.3: A cake is to be divided among 7 players P1, P2, P3, P4, P5, P6, P7 using the last diminisher method. The players play in the fixed order listed previously. In round 1, P1 cuts a piece and P2, P4, and P6 are the only diminishers. In round 2, P5 is the only diminisher. In round 3, every remaining player is a diminisher. In round 4, there are no diminishers.

a) Who gets the piece at the end of round 1?
   \[ P6 \] (the last diminisher)

b) Who cuts the piece at the beginning of round 2?
   \[ P1 \]

c) Who gets the piece at the end of round 2?
   \[ P5 \] (the last diminisher)

d) Who cuts the piece at the beginning of round 3?
   \[ P1 \]

e) Who gets a piece at the end of round 3?
   \[ P7 \] (the last diminisher)

f) Who cuts at the beginning of round 4?
   \[ P1 \]

g) Who gets a piece at the end of round 4?
   \[ P1 \]

h) How many rounds until everyone has a piece of cake?
   \[ 6 \]
   (one less than \# of players)

Example E4.
An island is to be divided among seven players (P1, P2, P3,...) using the last diminisher method. The players play in a fixed order with P1 first and P7 last. P3 gets his fair share at the end of round 1, and P7 gets his fair share at the end of round 3. There are no diminishers in rounds 2, 4, and 5.

a) Who is the last diminisher in round 1?
   \[ P3 \]

b) Which players gets a fair share at the end of round 2?
   \[ P3 \]

c) Which player cuts at the beginning of round 3?
   \[ P3 \]

d) Which player gets a fair share at the end of round 4?
   \[ P2 \]

e) Which player gets a fair share at the end of round 5?
   \[ P4 \]

f) Which player is the chooser in the final round?
   \[ P6 \]
3.6 The Method of Sealed Bids

Example F1 (#53 p114) Three sisters (Ana, Belle, and Chloe) wish to use the method of sealed bids to divide up 4 pieces of furniture that they shared as children. Their bids on each of the items are given in the table below. Describe the final outcome of this fair-division problem.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dresser</td>
<td>$150</td>
<td>$300</td>
<td>$275</td>
</tr>
<tr>
<td>Desk</td>
<td>180</td>
<td>150</td>
<td>165</td>
</tr>
<tr>
<td>Vanity</td>
<td>170</td>
<td>200</td>
<td>260</td>
</tr>
<tr>
<td>Tapestry</td>
<td>400</td>
<td>250</td>
<td>500</td>
</tr>
<tr>
<td>Total</td>
<td>900</td>
<td>900</td>
<td>1200</td>
</tr>
<tr>
<td>F/S</td>
<td>300</td>
<td>300</td>
<td>400</td>
</tr>
</tbody>
</table>

A: Desk (worth $180)
- Takes $120 from estate
- Gets $80 from surplus

B: Dresser (worth $300)
- Takes $0 from estate
- Gets $80 from surplus

C: Vanity + Tapestry ($760)
- Pays $360 to estate
- Gets $80 from surplus

Step 1: Determine Each Fair Share

Bidding: Each person gives an honest dollar assessment of each item.
Determine each player’s fair share = \( 1/N \) (total)

Step 2: First Settlement

- Each item goes to highest bidder.
- Payment to/from the estate

Step 3: Dividing the Surplus

If there is leftover money in the escrow account, it is to be divided and distributed evenly between all players.

Step 4: Final Settlement

List any items received and net amount of money received from or paid to the estate.
Conditions for Sealed Bids

1) Each player must have enough money to play the game
2) Each player must be willing to accept money

Example F2: Grandma leaves a house, a Rolls Royce car, and a Picasso painting to her children: Art, Betty, Carla, & Dave with the stipulation that the items cannot be sold. The bids are listed in the table below. Describe the final outcome of this fair division problem.

<table>
<thead>
<tr>
<th>Grandma’s Estate</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>House</td>
<td>220,000</td>
<td>250,000</td>
<td>211,000</td>
<td>198,000</td>
</tr>
<tr>
<td>Car</td>
<td>40,000</td>
<td>30,000</td>
<td>47,000</td>
<td>52,000</td>
</tr>
<tr>
<td>Painting</td>
<td>280,000</td>
<td>240,000</td>
<td>234,000</td>
<td>190,000</td>
</tr>
<tr>
<td>Total</td>
<td>540 K</td>
<td>520 K</td>
<td>492 K</td>
<td>440 K</td>
</tr>
</tbody>
</table>

Fair

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>House</td>
<td>135,000</td>
<td>130,000</td>
<td>123,000</td>
<td>110,000</td>
</tr>
</tbody>
</table>

A: Painting ($280,000)
  + pays $145,000 to estate
  + gets $21,000 from surplus

B: House ($250,000)
  + pays $120,000 to estate
  + gets $21,000 from surplus

C: No items
  + takes $123,000 from estate
  + gets $21,000 from surplus

D: Car ($52,000)
  + takes $58,000 from estate
  + gets $21,000 from surplus

FINAL

A: Painting + Pays $124,000 to estate
B: House + Pays $99,000 to estate
C: $144,000 (No items)
D: Car + $79,000

escrow
+ 145,000
+ 120,000
- 123,000
- 58,000

$84,000 surplus
\[ \div 4 \]
$21,000 back to each
3.7 The Method of Markers

Basic Idea: We line up all of our items in a random, but fixed sequence (an array). Each player breaks a group of items into N “equal” shares using “markers.” We try to assign everyone one of their sets of items.

Step 1: Bidding
Each player places a certain number of markers, separating the goods into equal, fair shares. #markers = 1 less than # of players

Step 2: Allocation
1. Locate the 1st marker: From this point to the left is that player’s share. Eliminate all his markers. He has his share!

2. Locate the 1st “2nd-marker”. From this point left to that player’s “1st-marker” is that player’s share. Eliminate all his markers. He has his share!

3. Repeat this process (1st “3rd-marker” left to that players “2nd marker” etc.)

4. Last player gets from their last marker to right.

Step 3: Dividing the leftovers
1. Draw straws to pick leftovers
2. If more leftovers than players, repeat step 1 and 2 with leftovers.

Example G.1 Four players A, B, C, D decide to divide a batch of candy using the method of markers. The placement of their markers is shown below. Describe the outcome of this fair division problem.

A: 9,10,11,12
B: 5,6
C: 1,2,3
D: 14,15

leftovers: 4,7,8,13
**Example G2:**

Ann, Bob, Charlie, and Dana decide to split 16 items using the method of markers.
Ann places her markers immediately to the right of items 4, 6, and 10
Bob places his markers immediately to the right of items 5, 10, and 13.
Charlie places his markers immediately to the right of items 3, 8, and 12.
Dana places her markers immediately to the right of items 4, 8, and 11.

Who gets item 4? leftover
Who gets item 9? D
Who gets item 14? B
Who gets item 7? leftover
Who gets item 11? D
Who gets item 15? B

**Example G3:** Three brothers: Xavier, Yoshi, and Zackius get a box of comic books as a gift. They decide to split the 9 comic books using the method of markers. Xavier likes Batman and X-Men comics, Yoshi likes Spiderman comics, and Zackius likes to read all types of comics. They lay the comics out on the floor as shown below.

![Comic Book Layout Image]

- a) What would Xavier’s bid look like?
  \[ B + \chi = 6 \div 3 = 2 \text{ each set} \]
- b) What would Yoshi’s bid look like?
  \[ S = 3 \div 3 = 1 \text{ each set} \]
- c) What would Zackius’ bid look like?
  \[ B + \chi + S = 9 \div 3 = 3 \text{ each set} \]
- d) Describe a fair division.
  \[ \chi: 2 \text{ Batman's} \]
  \[ y: 1 \text{ Superman} \]
  \[ z: 1 \text{ each type} \]

leftovers 4 Spiderman
2 X-men